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UNOBSERVED COMPONENT MODELS WITH ASYMMETRIC CONDITIONAL VARIANCES

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Abstract

In this paper, unobserved component models with GARCH disturbances are extended to allow for asymmetric responses of conditional variances to positive and negative shocks. The asymmetric conditional variance is represented by a member of the QARCH class of models. The proposed model allows to distinguish whether the possibly asymmetric conditional heteroscedasticity affects the short run or the long-run disturbances or both. We analyse the statistical properties of the new model and derive the asymptotic and finite sample properties of a QML estimator of the parameters. We propose to identify the conditional heteroscedasticity using the correlogram of the squared auxiliary residuals. Its finite sample properties are also analysed. Finally, we ilustrate the results fitting the model to represent the dynamic evolution of daily series of financial returns and gold prices, as well as of monthly series of inflation. The behaviour of volatility in both types of series is different. The conditional heteroscedasticity mainly affects the short run component in financial returns while in the inflation series, the heteroscedastic ity appears in the long-run component. We find asymmetric effects in both types of variables.

Keywords: Auxiliary residuals, financial series, GARCH, inflation, leverage effect, QARCH, structural time series models.

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1 Introduction

Economic time series can often be decomposed into components that have a direct interpretation, for example, trend, seasonal and transitory components; see Harvey (1989) for a detailed description of unobserved component models. In the simplest case, the series of interest, y_t, can be decomposed in a long-run component, representing an evolving level, μ_t, and a transitory component, ε_t. If the level follows a random walk and the transitory component is white noise, the resulting model is given by

$$\begin{aligned} y_t &= \mu_t + \varepsilon_t \\ \mu_t &= \mu_{t-1} + \eta_t \end{aligned} \tag{1}$$

where ε_t and η_t are mutually independent Gaussian white noise processes with variances h and q respectively. Model (1), known as random walk plus noise, has been very useful to represent the dynamic dependence of a large number of economic time series; see, for example, Durbin and Koopman (2001) for a recent reference containing several applications concerning this model.

The random walk plus noise model was extended by Harvey *et al.* (1992) to allow the variances of both, the short and the long-run components, to evolve over time following GARCH(1,1) models. In particular, the disturbances are defined by $\varepsilon_t = \varepsilon_t^{\dagger} h_t^{1/2}$ and $\eta_t = \eta_t^{\dagger} q_t^{1/2}$where $\varepsilon_t^{\dagger}$ and $\eta_t^{\dagger}$ are mutually independent Gaussian white noise processes and h_t and q_t are given by

$$\begin{aligned} h_t &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 h_{t-1} \\ q_t &= \gamma_0 + \gamma_1 \eta_{t-1}^2 + \gamma_2 q_{t-1} \end{aligned} \tag{2}$$

where the parameters α_0, α_1, α_2, γ_0, γ_1 and γ_2 satisfy the usual conditions to guarantee the positivity and stationarity of h_t and q_t.

Model (1) with the variances defined as in (2) is a Structural ARCH (STARCH) model. The main attractive of STARCH models is that they are able to distinguish whether the ARCH effects appear in the permanent and/or in the transitory component[1]. Unobserved component models with GARCH disturbances have been applied in

[1] Ord *et al.* (1997) propose an alternative unobserved component model with heteroscedastic errors where, instead of considering different disturbance processes for each component, the source of randomness is unique.

fields like macroeconomics and finance. For example, Evans and Wachtel (1989), Ball and Cechetti (1990) and Evans (1991) analyze inflation, Kim (1993) analyzes inflation and interest rates, Fiorentini and Maravall (1996) analyze the Spanish money supply and Bos *et al.* (2000) study series of returns.

The variances in equations (2) are specified in such a way that their responses to positive and negative changes in the corresponding disturbances are symmetric. However, in some cases, the empirical evidence suggests that the conditional variance may have a different response to shocks of the same magnitude but different sign. This phenomenon, known as *"leverage effect"* in the Financial Econometrics literature, has often been observed in high frequency financial data; see, for example, Shephard (1996) and the references therein. In the context of macroeconomic time series, Brunner and Hess (1993) point out the importance of considering the *"leverage effect"* in the modelization of inflation.

There are several alternative models proposed in the literature to represent asymmetric responses of volatility to positive and negative shocks; see Hentschel (1995) and He and Teräsvirta (1999) for two asymmetric models that encompass many of the most popular alternatives. In this paper, we consider the Generalized Quadratic ARCH (GQARCH) model originally proposed by Sentana (1995) because of its tractability. If the disturbances ε_t and η_t follow GQARCH(1,1) processes, their variances are given by,

$$\begin{aligned} h_t &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta \varepsilon_{t-1} + \alpha_2 h_{t-1} \\ q_t &= \gamma_0 + \gamma_1 \eta_{t-1}^2 + \delta \eta_{t-1} + \gamma_2 q_{t-1} \end{aligned} \tag{3}$$

respectively. The parameters in (3) should be restricted for the variances to be positive. In particular α_0, α_1, $\alpha_2 > 0$ and $\beta^2 \leq 4\alpha_1\alpha_0$. Similar restrictions are imposed on γ_0, γ_1, δ and γ_2. On the other hand, ε_t is covariance stationary if $\alpha_1 + \alpha_2 < 1$. Similarly, if $\gamma_1 + \gamma_2 < 1$, η_t is covariance stationary; see He and Teräsvirta (1999). Notice that the covariance stationarity of ε_t and η_t does not depend on the parameters β and δ that measure the asymmetry.

Sentana (1995) analyzes the properties of the GQARCH(1,1) model and points out that it is very similar to the GARCH(1,1) model. For example, the GARCH(1,1) and GQARCH(1,1) models for ε_t, in equations (2) and (3) respectively have the same unconditional mean and variance equal to zero and $\sigma_\varepsilon^2 = \frac{\alpha_0}{1-\alpha_1-\alpha_2}$ respectively. Furthermore, in both models, the odd moments are zero, the series ε_t is uncorrelated and the cross-correlations between ε_t^2 and ε_{t-h} are zero for all $h \geq 2$. When $h = 1$, $Cov(\varepsilon_t^2, \varepsilon_{t-1}) = \beta\sigma_\varepsilon^2$ in the GQARCH(1,1) model and zero in the GARCH(1,1) model. Using the results of He and Teräsvirta (1999), it is possible to derive the following expressions for the kurtosis of ε_t and autocorrelation function (acf) of ε_t^2

$$\kappa_\varepsilon = \frac{3(1-(\alpha_1+\alpha_2)^2)}{(1-3\alpha_1^2-\alpha_2^2-2\alpha_1\alpha_2)} + 3\frac{A^*}{(1-3\alpha_1^2-\alpha_2^2-2\alpha_1\alpha_2)} \tag{4}$$

$$\rho_{\varepsilon^2}(\tau) = \begin{cases} \frac{2\alpha_1(1-\alpha_1\alpha_2-\alpha_2^2)+A^*(3\alpha_1+\alpha_2)}{2(1-2\alpha_1\alpha_2-\alpha_2^2)+3A^*}, & \tau = 1 \\ \\ (\alpha_1+\alpha_2)^{\tau-1}\rho_{\varepsilon^2}(1), & \tau > 1 \end{cases} \tag{5}$$

where $A^* = (\beta/\sigma_\varepsilon)^2$. Notice that the kurtosis of ε_t is larger than in the symmetric GARCH model. For example, if $\alpha_0 = 0.05$, $\alpha_1 = 0.15$, $\alpha_2 = 0.8$ and $\beta = 0$, the kurtosis is 5.57 while if, for the same parameter values, $|\beta| = 0.1$, the kurtosis is 6.14. On the other hand, the autocorrelation function of the squares of a GQARCH(1,1) model decays at the same rate as in the GARCH(1,1) model. Furthermore, if β is small relative to σ_ε^2, as it is usually the case in empirical applications, the autocorrelation of order one is almost the same in both models. For example, for the same parameter values considered before, if $\beta = 0$, then $\rho_{\varepsilon^2}(1) = 0.3$ while if $|\beta| = 0.1$, then $\rho_{\varepsilon^2}(1) = 0.31$. Therefore, it seems that incorporating the leverage effect into the conditional variance increases the kurtosis of the process without increasing the autocorrelations of squares.

The asymmetry of the GQARCH(1,1) model is reflected in the corresponding "News Impact Curve" that is a shifted parabola. Consequently, these models pick up asymmetric effects in the presence of small shocks while models with a rotated parabola will capture the effects of large ones[2]. Furthermore, GQARCH models pick up the *"leverage effect"* in an additive way. Consequently, the estimation of these models is easier than in models that use a multiplicative specification like, for example, the EGARCH model of Nelson (1991).

[2] Hentschel (1995) point out that a model combining these two aspects in the *"News Impact Curve"* may give as a result either a cancellation of the asymmetric effect or an overestimation.

The objective of this paper is to extend the STARCH model by allowing the variances of the disturbances ε_t and η_t to follow GQARCH models. Hereafter, we call this new family of models Quadratic STARCH (Q-STARCH). These models are able to represent asymmetric responses of conditional variances to positive and negative disturbances distinguishing whether the asymmetry appears in the short or in the long-run components. Secondly, we will show how the autocorrelations of the squared auxiliary residuals corresponding to the long-run and transitory components can be used to identify which of these components is conditionally heteroscedastic.

The paper is organized as follows. Section 2 introduces the Q-STARCH model and describes its properties. Section 3 contains finite sample properties of the autocorrelations of squared observations and squared auxiliary residuals, which are useful to identify the presence of heteroscedastic asymmetric variances. In section 4, we analyze the asymptotic properties of a Quasi-Maximum Likelihood (QML) estimator of the parameters of the Q-STARCH model based on the prediction error decomposition of the Gaussian log-likelihood, while in section 5 we study its finite sample properties. In section 6, the Q-STARCH model is fitted to daily gold prices and financial returns and to monthly series of inflation. Finally, section 7 concludes the paper.

2 Q-STARCH model

In this section, we analyze the statistical properties of the Q-STARCH model defined by equations (1) and (3). Although the random walk plus noise model with GQARCH (1,1) disturbances is not stationary, it is possible to obtain a stationary series by taking first differences. Therefore, the stationary form is given by

$$\Delta y_t = \eta_t + \Delta \varepsilon_t. \tag{6}$$

From (6) it can be easily seen that y_t follows an ARIMA(0,1,1) with non-Gaussian innovations; see Harvey (1989). Furthermore, notice that the innovations of this model are uncorrelated although not independent; see Breidt and Davis (1992). The mean, variance and autocorrelation function of Δy_t are the same as in the homoscedastic random walk plus noise model; see, for example, Harvey (1989). The presence of asymmetric ARCH effects is reflected in the kurtosis of Δy_t given by

$$\begin{aligned}\kappa(\Delta y_t) &= \frac{3}{(q+2)^2}\{4q+q^2\frac{1-(\gamma_1+\gamma_2)^2+B^*}{1-3\gamma_1^2-2\gamma_1\gamma_2-\gamma_2^2} \\ &\quad + \frac{4(1-(\alpha_1+\alpha_2)^2+\alpha_1(1-\alpha_1-\alpha_1\alpha_2-\alpha_2^2))+2A^*(1+3\alpha_1+\alpha_2)}{1-3\alpha_1^2-2\alpha_1\alpha_2-\alpha_2^2}\}.\end{aligned} \tag{7}$$

where $q=\frac{\sigma_\eta^2}{\sigma_\varepsilon^2}$, $\sigma_\eta^2=\frac{\gamma_0}{1-\gamma_1-\gamma_2}$ and $B^*=(\frac{\delta}{\sigma_\eta})^2$. Notice that, in the homoscedastic case, when $\alpha_1=\gamma_1=\beta=\delta=0$, the kurtosis is, as expected, 3. The presence of ARCH effects, $\alpha_1\neq 0$ or $\gamma_1\neq 0$, causes excess kurtosis. Besides, in the asymmetric case, when $\beta\neq 0$ or $\delta\neq 0$, the excess kurtosis is even greater. Therefore, the kurtosis of a Q-STARCH model is bounded from bellow by the kurtosis of a symmetric STARCH model independently of the sign of β or δ. The kurtosis is, in general, a complicated function of the signal to noise ratio, q, and of the ARCH parameters. For example, assuming that the ARCH effects of the long and short run disturbances are identical, i.e. $\gamma_0=\alpha_0$, $\gamma_1=\alpha_1$ and $\gamma_2=\alpha_2$, the kurtosis increases more when the asymmetry of the transitory component increases (β) than when the asymmetry of the long-run disturbance increases (δ).

Furthermore, the skewness of Δy_t is given by

$$SK(\Delta y_t)=\frac{-3\beta}{\sigma_\varepsilon\,(q+2)^{3/2}}. \tag{8}$$

Note that only the asymmetry of the transitory component, β, affects the skewness coefficient. Looking at expressions (7) and (8), it seems that, indenpendently of the signal to noise ratio, q, the asymmetry of the transitory noise is more influential than the asymmetry of the long-run noise on the statistical properties of Δy_t. In any case, the main dynamic properties of (Δy_t) appear in the squares. After some tedious algebra, it is possible to derive the following expression of the autocovariance function of $(\Delta y_t)^2$

$$\gamma_{(\Delta y_t)^2}(\tau)=\begin{cases}\sigma_\varepsilon^4(q+2)^2(\kappa(\Delta y_t)-1), & \tau=0\\ \sigma_\varepsilon^4\{q^2(\kappa_\eta-1)\rho_{\eta^2}(1)+(\kappa_\varepsilon-1)[1+(2+\alpha_1+\alpha_2)\rho_{\varepsilon^2}(1)]\} & \tau=1\\ (\alpha_1+\alpha_2)\gamma_{(\Delta y_t)^2}(\tau-1)+[(\gamma_1+\gamma_2)-(\alpha_1+\alpha_2)](\gamma_1+\gamma_2)^{\tau-2}\gamma_{\eta^2}(1), & \tau\geq 2\end{cases} \tag{9}$$

where $\kappa(\Delta y_t)$ is given in (7) and $\rho_{\eta^2}(1)$ and $\gamma_{\eta^2}(1)$ are the acf and autocovariance of order one of η_t^2, respectively. From (9), it is straightforward to obtain the acf of $(\Delta y_t)^2$. Notice that the decay in the correlogram of the squared first differences is the same as for the symmetric STARCH model. Furthermore, when the persistence of the variances of the short and long-run components is similar, the decay of the autocorrelations is exponential with parameter $\alpha_1 + \alpha_2$. As expected, in the homoscedastic case, when $\alpha_1 = \gamma_1 = \beta = \delta = 0$, all the autocorrelations for $(\Delta y_t)^2$ are zero for lags greater than one and the autocorrelation at lag one is $\left(\frac{1}{q+2}\right)^2 = \left[\rho_{\Delta y_t}(1)\right]^2$, where $\rho_{\Delta y_t}(1)$ is the lag one autocorrelation of Δy_t. Therefore, the autocorrelations of the squared observations are equal to the squared autocorrelations of the row observations; see Maravall (1983). As an illustration, Figure 1 plots the acf of the squared first differences of three Q-STARCH models with parameters $\{\alpha_0 = 0.05,\ \alpha_1 = 0.15,\ \alpha_2 = 0.8, \gamma_0 = 0.05, \gamma_1 = \gamma_2 = 0\}$, $\{\alpha_0 = 0.05, \alpha_1 = \alpha_2 = 0, \gamma_0 = 0.05,\ \gamma_1 = 0.15\ ,\ \gamma_2 = 0.8\}$ and $\{\alpha_0 = \gamma_0 = 0.05,\ \alpha_1 = \gamma_1 = 0.15,\ \alpha_2 = \gamma_2 = 0.8\}$ respectively. Given that, as we have seen before, the presence of asymmetries only affects slightly the autocorrelations of squares, we have fixed $\beta = \delta = 0$ in all models. In Figure 1, it is possible to observe that the shape of the acf of squares depends on whether the conditional heteroscedasticity affects the short-run, the long-run or both components.

The information about the asymmetric response of the variances to positive and negative innovations is more evident in the cross-correlations between $(\Delta y_t)^2$ and $(\Delta y_{t-\tau})$ that are given by

$$Corr\left[(\Delta y_t)^2, (\Delta y_{t-\tau})\right] = \begin{cases} 0, & \forall\, \tau < -1 \\ \frac{2\beta}{\sigma_\varepsilon (q+2)^{3/2}(\kappa(\Delta y_t)-1)^{1/2}}, & \tau = -1 \\ \frac{-3\beta}{\sigma_\varepsilon (q+2)^{3/2}(\kappa(\Delta y_t)-1)^{1/2}} & \tau = 0 \\ \frac{\delta q - \beta(\alpha_1+\alpha_2)}{\sigma_\varepsilon (q+2)^{3/2}(\kappa(\Delta y_t)-1)^{1/2}}, & \tau = 1 \\ \frac{\delta(\gamma_1+\gamma_2)^{\tau-1} q + \beta((\alpha_1+\alpha_2)^{\tau-2} - (\alpha_1+\alpha_2)^{\tau})}{\sigma_\varepsilon (q+2)^{3/2}(\kappa(\Delta y_t)-1)^{1/2}} & \tau \geq 2 \end{cases} \qquad (10)$$

Note that in the symmetric STARCH case these third order moments are always zero.

Figure 2 plots the cross-correlation function in (10) for the same Q-STARCH models previously considered in Figure 1 with the parameters $\beta = 0.17$ in the first and third models and $\delta = 0.17$ in the second and third models. It is possible to observe that the shape of the cross-correlograms depends on whether the asymmetry appears in the transitory, in the long-run or in both components. In the first case, when the asymmetry only appears in the transitory component, all the cross-correlations are zero except for lags between -1 and 1. However, when the permanent component follows a GQARCH(1,1) model, there is an exponential decay of the cross-correlations of negative order towards zero. Finally, if both components have similar asymmetric effects, the dominant effect is the corresponding to the transitory component. In general, the magnitude of the cross-correlations is so small that they are not an useful instrument to identify the presence of asymmetries in the variances of unobserved component models.

In unobserved component models, it can also be useful to analyze the auxiliary residuals, that estimate the disturbances of each component; see Maravall (1987) and Harvey and Koopman (1992). The latter authors show that the Minimum Mean Square Linear (MMSL) estimators of ε_t and η_t are given by

$$\widehat{\eta}_t = \frac{(1+\theta)^2 \Delta y_t}{(1-\theta L)(1+\theta F)} \tag{11}$$

$$\widehat{\varepsilon}_t = \frac{\theta}{1+\theta^2}(\widehat{\eta}_{t+1} - \widehat{\eta}_t) \tag{12}$$

where F is the lead operator such that $Fx_t = x_{t+1}$, L is the lag operator such that $Lx_t = x_{t-1}$ and θ is the moving average parameter of the reduced form of Δy_t given by $\theta = \frac{-q-2+\sqrt{q^2+4q}}{2}$. Harvey and Koopman (1992) show that, if time is reversed, $\widehat{\eta}_t$ follows an AR(1) model with parameter θ whereas $\widehat{\varepsilon}_t$ follows a strictly noninvertible ARMA(1,1) process with autoregressive parameter θ. The first order autocorrelation of $\widehat{\varepsilon}_t$ is then given by $\rho_{\widehat{\varepsilon}}(1) = -0.5(\theta+1)$.

Finally, the variances of $\widehat{\eta}_t$ and $\widehat{\varepsilon}_t$ are given by

$$Var(\widehat{\eta}_t) = \frac{(-\theta q)^2(q+2)\sigma_\varepsilon^2}{(1-\theta^4)} \tag{13}$$

$$Var(\widehat{\varepsilon}_t) = -\frac{2\theta\sigma_\varepsilon^2}{1-\theta} \tag{14}$$

3 Finite sample properties of autocorrelations of squares

The properties of the Q-STARCH model described before suggest to identify the presence of conditional heteroscedasticity by using the correlogram of $(\Delta y_t)^2$ as well as the corresponding correlogram of the squared auxiliary residuals.

In this section, the finite sample properties of these correlations in the random walk plus noise model with GQARCH disturbances are analyzed by means of extensive Monte Carlo experiments. The series have been generated with sample sizes $T = 300$, $T = 1000$ and $T = 3000$, by the following four Q-STARCH models[3]

	α_0	α_1	α_2	β	γ_0	γ_1	γ_2	δ
$M1$	0.25	0	0	0	0.05	0.15	0.8	-0.17
$M2$	0.05	0.15	0.8	-0.17	4.0	0	0	0
$M3$	0.05	0.15	0.8	-0.17	0.2	0.15	0.8	-0.17
$M4$	4.0	0	0	0	0.05	0.15	0.8	-0.17
$M5$	0.05	0.15	0.8	-0.17	0.25	0	0	0
$M6$	0.2	0.15	0.8	-0.17	0.5	0.15	0.8	-0.17

The first three models have $q = 4.0$, while $q = 0.25$ for the rest of the models. Models $M1$ and $M4$ have an homoscedastic short-run noise while the long-run component is heteroscedastic. On the other hand, the short-run disturbances of models $M2$ and $M5$ are heteroscedastic while the long-run variances are constant. In $M3$ and $M6$ both components are conditionally heteroscedastic. The asymmetry parameter -0.17 has been chosen as it is the largest to guarantee the positivity of the conditional variances.

For each model, we generate 1000 replicates and for each replicate, we compute the sample autocorrelations of $(\Delta y_t)^2$, $\widehat{\varepsilon}_t^2$ and $\widehat{\eta}_t^2$ for lags up to 36. Then, we compute the average mean and standard deviation of the estimates through all replicates[4].

The Monte Carlo results on the estimated $\rho_{(\Delta y_t)^2}(\tau) = Corr\left[(\Delta y_t)^2, (\Delta y_{t-\tau})^2\right]$ have been reported in Table 1 for $\tau = 1$ and 10. Correlations of Δy_t have a large

[3]Results for other designs are not reported here to save space but are available from the authors upon request.

[4]All simulations have been carried out on a Pentium desktop computer using our own FORTRAN codes.

negative bias. The bias is larger in those models with $q=4$, than when $q=0.25$ and decreases with the sample size. On the other hand, the empirical standard deviation decrease with the sample size at an approximate rate of $\sqrt{T}$ while for $M1$, $M2$ and $M3$, in which $q=4$, this rate is lower.

Table 1 also reports Monte Carlo results on the sample autocorrelations of the squares of $\widehat{\varepsilon}_t^2$ and $\widehat{\eta}_t^2$. In this case, same conclusion can be obtained about the empirical standard deviation decrease, that is lower than $\sqrt{T}$.

The results are illustrated in Figure 3, that plots in the first row, for models M1, M2,M4 and M5, the mean autocorrelation function of $(\Delta y_t)^2$ when $T=1000$ together with the corresponding acf derived in previous section. In this Figure, it can be observed that the bias is huge specially when q is large and the transitory component is conditionally heteroscedastic or when q is small and the conditional heteroscedasticity appears in the long-run noise. In these cases, it seems that the sample autocorrelations of $(\Delta y_t)^2$ are not useful to identify the presence of conditionally heteroscedastic unobserved noises. The second and third rows of Figure 3 plot the mean of the sample autocorrelations of the squared auxiliary residuals, $\widehat{\varepsilon}_t$ and $\widehat{\eta}_t$, together with the corresponding acf's obtained assuming homoscedasticity. First, notice that the autocorrelations are larger than expected if the corresponding component were homoscedastic. Therefore, the autocorrelations of squared auxiliary residuals can be a useful instrument to detect conditional heteroscedasticity. Furthermore, in the first two models, the autocorrelations of squares are larger in $\widehat{\eta}_t$ than in $\widehat{\varepsilon}_t$. On the other hand, for the last two models, the autocorrelations of $\widehat{\varepsilon}_t^2$ are larger than the autocorrelations of $\widehat{\eta}_t^2$. Notice that, this is a rather useful result because it allows to identify the component that is conditionally heteroscedastic. Finally, it is also important to notice that, as expected, when the transitory noise, ε_t, is heteroscedastic, the autocorrelations of $\widehat{\varepsilon}_t^2$ are larger the smaller is q. However, when the conditional heteroscedasticity affects the long-run noise, η_t, the autocorrelations of $\widehat{\eta}_t^2$ are larger the larger is q.

It may also seem rather natural to use the cross-correlations of the auxiliary residuals to conclude whether the conditional variances of the noises of the unobserved components are asymmetric. However, as we have pointed out before for Δy_t, the magnitudes of the cross-correlations are so small that they are not going to be useful in that sense. Consequently, it is not worth to try to derive the analytical expressions of the corresponding cross-correlations.

4 Estimation of Q-STARCH model

Harvey *et al.* (1992) proposed a QML estimator of the parameters of the STARCH model based on expressing the local level model in an augmented state space form. The state vector is augmented by lags of μ_t in such a way that it is possible to get estimations of both disturbances and their associated correction factors. The measurement and transition equations are respectively given by

$$y_t = \mu_t + \varepsilon_t = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \alpha_t + \varepsilon_t$$

$$\alpha_t = \begin{bmatrix} \mu_t \\ \mu_{t-1} \\ \eta_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_{t-1} \\ \mu_{t-2} \\ \eta_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \eta_t. \tag{15}$$

Even if $\varepsilon_t^\dagger$ and $\eta_t^\dagger$ are assumed to be Gaussian processes, STARCH models are not conditionally Gaussian, since knowledge of past observations does not imply knowledge of past disturbances. Consequently, the QML estimator is based on treating the model as if it were conditionally Gaussian and running the Kalman filter to obtain the one-step ahead prediction errors and their variances to be used in the expression of the Gaussian likelihood given by

$$\log L = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log F_t - \frac{1}{2}\sum_{t=1}^{T}\frac{\nu_t^2}{F_t}, \tag{16}$$

where ν_t, $t = 1, ..., T$ are the innovations and F_t their corresponding variances. The QML estimator, $\widehat{\Psi}$, is obtained by maximizing the Gaussian likelihood in (16) with respect to the unknown parameters. Harvey *et al.* (1992) give a detailed description of the Kalman filter for the random walk plus noise model with GARCH disturbances.

In this section, we extend the QML estimator proposed by Harvey *et al.* (1992) to the estimation of the random walk plus white noise model with GQARCH(1,1) disturbances. Estimation of GQARCH models is easier using the following reparametrization proposed by Sentana (1995) to guarantee the positivity of the variances h_t and q_t.

$$h_t = a_0 + a_1^2(\varepsilon_{t-1} - b)^2 + a_2^2 h_{t-1}$$

$$q_t = g_0 + g_1^2(\eta_{t-1} - d)^2 + g_2^2 q_{t-1} \tag{17}$$

where the parameters of interest are $\alpha_0 = a_0 + a_1^2 b^2$, $\alpha_1 = a_1^2$, $\alpha_2 = a_2^2$ and $\beta = -2ba_1^2$. Similar transformations apply to the parameters of q_t. After estimating the parameter vector, $\Psi = (a_0, a_1, a_2, b, g_0, g_1, g_2, d)$, these transformations can be used to obtain the original parameters of the model.

When the disturbances are GQARCH processes, some of the equations of the Kalman filter should be modified. In particular, the filter requires expressions of the following estimates of ε_t and η_t

$$\begin{aligned} \hat{\varepsilon}_t &= y_t - m_t, \\ \widehat{\eta}_t &= m_t - m_{t-1|t}, \end{aligned} \tag{18}$$

where $m_t = \underset{t}{E}\ \mu_t$ and $m_{t-1|t} = \underset{t}{E}\ \mu_{t-1}$ are MMSL updated estimates of μ_t and μ_{t-1} obtained in a natural way by the augmentation of the state vector by μ_{t-1} in (15). The t under the expectation operator means that the expectation is conditional on the information available at time t. Note that there is no need to include ε_t in the state vector in order to get an expression of its estimate and corresponding variance. The filter also requires expressions of the conditional variances of the disturbances ε_t and η_t. For simplicity, we consider first the Q-STARCH model with the parameters α_2 and γ_2 fixed to zero. In this case, the conditional mean of ε_t is zero and its conditional variance is given by

$$H_t = \underset{t-1}{E}\ \varepsilon_t^2 = a_0 + a_1^2(\hat{\varepsilon}_{t-1} - b)^2 + a_1^2 P_{t-1} \tag{19}$$

where $P_t = \underset{t}{E}\ (\mu_t - m_t)$. Similarly, the conditional mean of the disturbance of the permanent component, η_t, is zero and its conditional variance is given by

$$Q_t = \underset{t-1}{E}\ \eta_t^2 = g_0 + g_1^2(\hat{\eta}_{t-1} - d)^2 + g_1^2 P_{t-1}^{\eta} \tag{20}$$

where $P_{t-1}^{\eta} = P_t + P_{t-1|t} - 2P_{t,t-1|t}$, $P_{t-1|t} = \underset{t}{E}\ (\mu_{t-1} - m_{t-1/t})^2$ and $P_{t,t-1/t} =$

$\underset{t}{E}\ (\mu_t - m_t)(\mu_{t-1} - m_{t-1/t})$. The required P_t, $P_{t-1|t}$ and $P_{t,t-1|t}$ are also provided by the Kalman filter.

In order to carry out the initialization of the filter, we set $m_1 = y_1$ and $P_1 = \underset{0}{E}\ \varepsilon_1 = \sigma_\varepsilon^2 = \frac{a_0 + a_1^2 b^2}{1 - a_1^2}$. In the framework of a random walk plus white noise this is equivalent to use a diffuse prior. Furthermore, if the conditional variance of η_t at time $t-1$ is also set equal to its unconditional variance, the Kalman filter can be started with $\underset{t-1}{E}\ (\varepsilon_2^2) = \sigma_\varepsilon^2$ and $\underset{t-1}{E}\ (\eta_2^2) = \sigma_\eta^2$.

If the parameters α_2 and γ_2 are different from zero, Harvey *et al.* (1992) suggest to consider the following alternative definitions of h_t and q_t

$$h_t = a_0 + a_1^2(\varepsilon_{t-1} - b)^2 + a_2^2 \underset{t-2}{E}\ (h_{t-1}),$$

$$q_t = g_0 + g_1^2(\eta_{t-1} - d)^2 + g_2^2 \underset{t-2}{E}\ (q_{t-1}). \tag{21}$$

Notice that $\underset{t-1}{E}\ (\varepsilon_t^2) = \underset{t-1}{E}\ (h_t)$ and $\underset{t-1}{E}\ (\eta_t^2) = \underset{t-1}{E}\ (q_t)$. Consequently, using equations (19) and (20), the following expressions are obtained

$$\begin{aligned} H_t &= a_0 + a_1^2(\hat{\varepsilon}_{t-1} - b)^2 + a_1^2 P_{t-1} + a_2^2 H_{t-1} \\ Q_t &= g_0 + g_1^2(\hat{\eta}_{t-1} - d)^2 + g_1^2 P_{t-1}^\eta + g_2^2 Q_{t-1}. \end{aligned} \tag{22}$$

In order to obtain the asymptotic distribution of the QML estimator, Harvey *et al.* (1992) suggest to consider that the variances h_t and q_t are given by equations (22). In this case, the Kalman filter is exactly the same as the one previously described but the model is conditionally Gaussian. Consequently, the filter and the likelihood in (16) are exact and the usual asymptotic theory can be applied. Under very general conditions, the asymptotic distribution of $\widehat{\Psi}$ can be approximated by a multivariate normal distribution with mean Ψ and covariance matrix $(Avar)^{-1}$. The ij'th element of the matrix $Avar$ is given by

$$IA_{ij}(\Psi) = \frac{1}{2} E \left[\sum_{t=1}^{T} \frac{1}{F_t^2} \frac{\partial F_t}{\partial \Psi} \frac{\partial F_t}{\partial \Psi'} + \sum_{t=1}^{T} \frac{1}{F_t} \frac{\partial \nu_t}{\partial \Psi} \frac{\partial \nu_t}{\partial \Psi'} \right]. \tag{23}$$

see, for example, Harvey (1989). The derivatives in expression (23) can be numerically evaluated as explained by Harvey (1989).

Once, the matrix $Avar$ has been computed, the delta method can be used to obtain the covariance matrix of the parameters of interest.

5 Finite sample properties of QML estimator

In this section, we analyze the finite sample properties of the QML estimator by means of Monte Carlo experiments. The series are simulated by the following alternative Q-STARCH models with parameters $(\alpha_0, \alpha_1, \alpha_2, \beta, \gamma_0, \gamma_1, \gamma_2, \delta)$ [5] given by

	α_0	α_1	α_2	β	γ_0	γ_1	γ_2	δ
$M1$	0.01	0.2	0	-0.05	0.01	0.1	0	-0.05
$M2$	0.01	0.2	0.5	-0.05	0.01	0.1	0.7	-0.05
$M3$	0.25	0	0	0	0.05	0.15	0.8	0
$M4$	0.05	0.15	0.8	0	4	0	0	0
$M5$	4	0	0	0	0.05	0.15	0.8	0
$M6$	0.05	0.15	0.8	0	0.25	0	0	0

The sample sizes considered are $T = 300$, 1000 and 3000. The numerical optimization of the likelihood has been performed using the IMSL subroutine DBCPOL with the parameters α_0 and γ_0 restricted to be nonnegative, and $\alpha_1 + \alpha_2$ and $\gamma_1 + \gamma_2$ restricted to be between 0 and 1.

Table 2 reports the Monte Carlo means and standard deviations (brackets) for models $M1$ and $M2$. This table also shows, in squared brackets, the corresponding approximated asymptotic standard deviation computed using expression (22). The results for model $M1$ show that, the biases of all the parameters are rather small even when $T = 300$. However, the asymptotic standard deviations of the ARCH parameters provide an adequate approximation to the empirical standard deviations only for the biggest sample size. In general, the asymptotic standard deviation is larger than the empirical standard deviation that decreases with the sample size at rate $\sqrt{T}$, approximately. Figure 4 plots kernel estimates of the densities of the parameter estimates of this model. This figure illustrates that the asymptotic Normal approximation of the QML estimator is adequate for relatively large sample sizes as, for example, $T = 3000$.

[5] Results for other parameter designs are available by the authors upon request.

The results for model $M2$ are, in general, similar to the previous ones. However, it is possible to observe that it seems to be a negative correlation between the parameters α_1 and α_2. The parameter α_1 is overestimated while α_2 is underestimated. The same effect can be observed with respect to the parameters γ_1 and γ_2. For example, when $T = 300$, the empirical correlations between α_1 and α_2 and between γ_1 and γ_2 are -0.57 and -0.61, respectively. When the sample size is $T = 3000$, these correlations are even bigger, -0.73 and -0.88 respectively. Notice that these high correlations could be expected since we are estimating imposing the stationarity restrictions, $\alpha_1 + \alpha_2 < 1$ and $\gamma_1 + \gamma_2 < 1$ and the parameters chosen are very close to these boundaries. On top of that, we can observe that the presence of the GARCH parameters worsens the estimation of the asymmetry parameter, specially if such asymmetry appears in the short run variance. Figure 5 plots the corresponding kernel densities for the parameters of model $M2$. It can be observed that the parameters of the variance of the transitory component are estimated with worse properties than the parameters of the long-run component. This is specially clear in the case of the estimates of the parameter α_2, which have rather unpleasant properties even when $T = 3000$.

To illustrate the problems faced when the Quasi-likelihood is maximized, Figure 6 plots the Gaussian likelihood in (16) for series simulated by Q-STARCH processes with asymmetry and conditional heteroscedasticity in the transitory component and four different specifications in the permanent one as a function of the parameters a_1 and b. Note that the function becomes flatter as the number of parameters increases. On the other hand, Figure 6 shows that the log-likelihood has local maximum, and consequently, the performance of any optimization algorithm strongly depends on the initial values provided. Finally, it is important to realize that the difficulties estimating the parameter $\alpha_1 = a_1^2$ could be due to the fact that the log-likelihood is rather flat when b is in its maximum.

Finally, Figure 7 plots kernel estimates of the densities of the Monte Carlo of the estimates of the parameters β and δ of models M3 to M6, which are actually zero. The main objective of these experiments is to analyze whether the sample distribution of the QML estimators of the parameters β and δ can be used to infer whether the transitory or the long-run conditional variances are asymmetric. Looking at the kernel densities plotted in Figure 7, it seems that the null hypothesis $H_0 : \beta = 0$ can be tested using standard results.

6 Empirical application

In this section we fit the Q-STARCH model to three daily financial series of returns, a daily series of gold prices and four monthly inflation series.

6.1 Daily series of gold prices and financial returns

In this subsection, we analyze empirically three financial time series of daily prices of the Nikkei 225 index observed from January 3, 1994 to December, 29, 2000 with a sample size of $T = 1825$[6] and of the Hewlett-Packard and Exxon stocks observed from January 3, 1994 to May 20, 2003 with $T = 2362$[7]. Finally, we also analyze a daily series of the logarithm of gold prices in US also observed from January 1, 1985 to December, 3, 1989 with $T = 1074$[8]. Several descriptive sample moments of the first differences of these series are reported in Table 3. Figure 8 plots the correlograms of Δy_t and $(\Delta y_t)^2$ for the four series. All the series show excess kurtosis and autocorrelations of squares larger than expected if they were linear.

The estimates of the parameters of the homoscedastic random walk plus noise model are shown in Table 4.[9] Note that in all cases $\hat{q}$ is rather large, meaning that in these series the variability of the permanent component dominates. Table 5 also shows several diagnostic statistics of the estimated innovations, ν_t, and the auxiliary residuals, $\hat{\varepsilon}_t$ and $\hat{\eta}_t$. In particular, for each of these series, we report the Box-Ljung statistics of order 10 for the original series and their squares. With respect to the innovations, the Box-Ljung statistic, $Q(10)$, does not show a strong evidence of autocorrelation. However, the corresponding statistic for the squares, $Q^2(10)$, is highly significant at any usual level. Consequently, the series of innovations may exhibit

[6] The series can be obtained from the Journal of Applied Econometrics data archive at: *http://qed.econ.queensu.ca/jae/*. The series is extracted from the file *index.data* belonging to Franses et al. (2002).

[7] Both Hewlett-Packard and Exxon series can be obtained from *http://finance.yahoo.com/*.

[8] The series can be obtained from: *http://www-personal.buseco.monash.edu.au/~hyndman/TSDL/*

[9] These estimates have been obtained using the program STAMP 6.20 of Koopman *et al.* (2000).

some kind of conditional heteroscedasticity. With respect to the auxiliary residuals, remember that they are serially correlated. For instance, the theoretical autocorrelation of order one for the Nikkei of $\widehat{\varepsilon}_t$ is $\rho_\varepsilon(1) = -0.4671$ and the theoretical acf of $\widehat{\eta}_t$ is $\rho_\eta(\tau) = 0.066\rho(\tau-1)$, $\tau = 2, 3...$, with $\rho_\eta(1) = 0.66$. Observe that the estimated autocorrelations in Table 4 are very close to their theoretical counterparts. If the noises were homoscedastic, the autocorrelations of the squared residuals are expected to be equal to the squared autocorrelations of the original residuals. However, in Table 4, it is possible to observe that the autocorrelations of squares are clearly larger than the squared autocorrelations, suggesting the presence of conditional heteroscedasticity. Finally, the Box-Ljung statistics for the squared residuals of the transitory, $\widehat{\varepsilon}_t$ and permanent component, $\widehat{\eta}_t$, reject clearly the null of homoscedasticity. Therefore, it seems that both components may be conditionally heteroscedastic.

The preferred Q-STARCH model consists in a GQARCH(1,1) model for the permanent component, and no ARCH effect in the transitory component disturbance, that is, a model of the form,

$$\begin{aligned} \hat{h}_t &= \hat{\alpha}_0 \\ \hat{q}_t &= \hat{\gamma}_0 + \hat{\gamma}_1\eta_{t-1}^2 + \hat{\delta}\eta_{t-1} + \hat{\gamma}_2\hat{q}_{t-1} \end{aligned} \tag{24}$$

Table 5 reports the estimation results, where values between brackets are t - statistics. Note that the estimate of δ is significant and negative for the Nikkei 225, Hewlett-Packard and Exxon, meaning that the only asymmetric effect is produced in the permanent component and that a negative shock affects more the conditional variance than a positive one. The estimate δ for the gold series is positive, that is, a positive shock affects the conditional variance more than a negative one.

6.2 Inflation time series

In this subsection, we analyze empirically monthly series of inflation corresponding to Japan and three European countries (Germany, Italy and United Kingdom). Inflation rates, y_t, are obtained as $y_t = (\log(CPI_t) - \log(CPI_{t-1})) \times 100$ where CPI stands for consumer price index. The European CPI were observed from January, 1962 to August, 2001 with $T = 476$, while for Japan the data were observed from January, 1970 to August, 2001, with $T = 380$ [10]. Intervention analysis and seasonal adjustment of all series were carried out with the program STAMP 6.20. Several descriptive sample moments of Δy_t are reported in Table 6. In these series the evidence of conditional heteroscedasticity is not so strong as in the daily series analyzed before.

The estimates of the parameters of the homoscedastic random walk plus noise model are reported in Table 7. Note that in all countries the estimated signal to noise ratio, $\widehat{q}$, is less than one, meaning that the estimated variance of the permanent component, $\hat{\sigma}^2_\eta$, is small compared with the variance of the transitory component, $\hat{\sigma}^2_\varepsilon$.

Table 8 also reports several diagnostic statistics of the estimated innovations and the auxiliary residuals. In particular, for each of these series, we report the estimated autocorrelations up to order 5 of the original and squared observations as well as the corresponding Box-Ljung statistics. With respect to the innovations, they may exhibit some kind of conditional heteroscedasticity in the cases of Italy and Japan. The same conclusion is reached looking at the Box-Ljung statistics for the squared residuals of the transitory component. Finally, notice that the autocorrelations of $\hat{\eta}_t^2$ are approximately equal to the squared autocorrelations of $\hat{\eta}_t$. Therefore, it seems that the long-run noises are not conditionally heteroscedastic while the transitory noises of Italy and Japan may have some kind of conditional heteroscedasticity.

For all countries the preferred Q-STARCH model consists in a GQARCH(1,1) model for the transitory component, and no ARCH effect in the permanent component disturbance given by

$$\begin{aligned} \hat{h}_t &= \hat{\alpha}_0 + \hat{\alpha}_1 \varepsilon_{t-1}^2 + \hat{\beta}\varepsilon_{t-1} + \hat{\alpha}_2 \hat{h}_{t-1} \\ \hat{q}_t &= \hat{\gamma}_0 \end{aligned} \tag{25}$$

The estimation results reported in Table 8, are in concordance with the conclusions derived from the analysis of the auxiliary residuals. The ARCH parameter α_1 is clearly significant for Italy and Japan while for Germany and UK is not statistically different from zero [11]. Therefore, the monthly inflation in Germany and UK seem to be homoscedastic, while in Italy and Japan the short-run component is conditionally heteroscedastic. However, the asymmetry parameter is significant in Japan at the

[10] The series can be obtained from the OECD Statistical Compendium, edition 02#2001.

[11] Notice that if $\alpha_1 = 0$, the parameters α_2 and β are not identified.

10% level. As this parameter is positive, it implies that when the short run inflation rises, the uncertainty associated with future inflation increases more than when it goes down. Therefore, our results support the Friedman hypothesis, according to which, a present positive shock in inflation will affect tomorrow's uncertainty about inflation more than a negative one; see Friedman (1977).

7 Summary and conclusions

In this paper we propose a new unobserved components model with conditionally heteroscedastic noises that allows the corresponding conditional variances to respond asymmetrically to negative and positive shocks. We denote this model as Q-STARCH. We show that the asymptotic distribution of a QML estimator could be an adequate approximation to the finite sample distribution. Consequently, inference can be based on classical results.

We also show how the autocorrelations of squared auxiliary residuals contain information useful to identify which of the components is conditionally heteroscedastic. However, the sample autocorrelations are severely biased towards zero making, in some cases, the identification of conditional heteroscedasticity a difficult task. In this sense, it may be useful to analyze the behavior of the portmanteau statistic proposed by Rodriguez and Ruiz (2003) to test the uncorrelatedness of a time series that takes into account not only the magnitude of the sample autocorrelations but also whether these autocorrelations have any systematic pattern.

Finally, we show with empirical examples how the Q-STARCH model can be useful for both financial and macroeconomic variables.

Two generalizations of the model are of special interest for the empirical applications: first, the extension to models with seasonal components so that the model can be directly implemented to analyze seasonal data as inflation, and second, the multivariate generalization. Further research is being carried out in these directions.

References

Ball, L. and S.G. Cecchetti (1990), Inflation and Uncertainty at Short and Long Horizons, *Brooking Papers on Economic Activity*, 215-254.

Bos C., R. Mahieu and H. van Dijk (2000), On the Variation of Hedging Decisions in Daily Currency Risk Management, *Econometric Institute Report EI 2000-20/A, Erasmus University Rotterdam.*

Breidt, F.J. and R.A. Davis (1992), Time-Reversibility, Identifiability and Independence of Innovations for Stationary Time Series, *Journal of Time Series Analysis*, 13, 377-390.

Brunner, A.D. and G.D. Hess (1993), Are Higher Levels of Inflation Less Predictable? A State Dependent Conditional Heteroskedasticity Approach, *Journal of Business & Economic Statistics*, vol. 11, 2, 187-197.

Durbin, J. and S.J. Koopman (2001), Time Series by State Space Methods, *Oxford University Press.*

Evans, M. (1991), Discovering the Link between Inflation Rates and Inflation Uncertainty, *Journal of Money, Credit and Banking*, 23,2, 169-184.

Evans, M. and P. Wachtel (1993), Inflation Regimes and the Sources of Inflation Uncertainty, *Journal of Money, Credit and Banking,* 25, 2*nd* part, 475-511.

Fiorentini G. and A. Maravall (1996), Unobserved Components in ARCH Models: An Application to Seasonal Adjustment, *Journal of Forecasting*, vol. 15, 175-201.

Franses, P.H., M. Leij and R. Paap (2002), Modelling and Forecasting Level Shifts in Absolute Returns, *Journal of Applied Econometrics*, 17, 5, 601-616.

Friedman, M. (1977), Nobel Lecture: Inflation and Unemployement, *Journal of Political Economy*, 85, 451-472.

Harvey, A.C. (1989), *Forecasting, Structural Time Series Models and the Kalman Filter*, Cambridge University Press.

Harvey, A.C. and S. Koopman (1992), Diagnostic Checking of Unobserved-Components Time Series Models, *Journal of Business & Economic Statistics*, 10, 377-389.

Harvey, A.C., E. Ruiz and E. Sentana (1992), Unobserved Component Time Series Models with ARCH Disturbances, *Journal of Econometrics,* 52, 129-157.

Koopman S.J., Harvey, A.C., Doornik, J.A. and Shephard, N. (2000). Stamp: Structural Time Series Analyser, Modeller and Predictor, London: Timberlake Consultants Press.

He, C. and T. Teräsvirta (1999), Properties of Moments of a Family of GARCH Processes, *Journal of Econometrics*, 92, 173-92.

Hentschel, L.F. (1995), All in the Family: Nesting Linear and non Linear GARCH Models, *Journal of Financial Economics*, 39, 139-64.

Kim, C.J. (1993), Unobserved-Component Time Series Models with Markov-Switching Heteroskedasticity: Changes in Regime and the Link between Inflation Rates and Inflation Uncertainty, *Journal of Business and Economic Statistics*, 11, 341-349.

Maravall, A. (1983), An Application of Nonlinear Time Series Forecasting, *Journal of Business and Economic Statistics*, 1, 66-74.

Maravall, A. (1987), Minimum Mean Squared Error Estimation of the Noise in Unobserved Component Models, *Journal of Business & Economic Statistics*, 5, 115-120.

Nelson, D.B. (1991), Conditional Heteroscedasticity in Asset Returns: a New Appoach, *Econometrica*, 59, 347-70.

Ord, J.K., A.B. Koehler and R.D. Snyder (1997), Estimation and Prediction for a Class of Dynamic Nonlinear Statistical Models, *Journal of the American Statistical Association*, 92, No. 440, 1621-1629.

Rodriguez, J. and E. Ruiz (2003), A Powerful Test for Conditional Heteroscedasticity for Financial Time Series with Highly Persistent Volatilities. Manuscript.

Sentana, E. (1995) Quadratic ARCH models, *Review of Economic Studies*, 62, 639-61.

Shephard, N. (1996), Statistical Aspects of ARCH and stochastic volatility, in O.E. Barndorff-Nielsen, D.R. Cox and D.V. Hinkley (eds.), *Statistical Models in Econometrics, Finance and other Fields*, London: Chapman & Hall, 1-67.

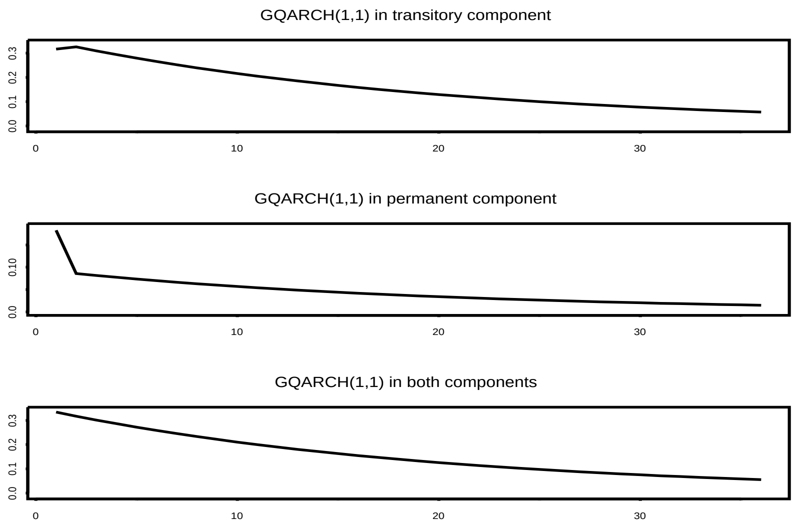


Figure 1: Autocorrelation function of $(\Delta y_t)^2$ of different Q-STARCH models with $q = 1$.

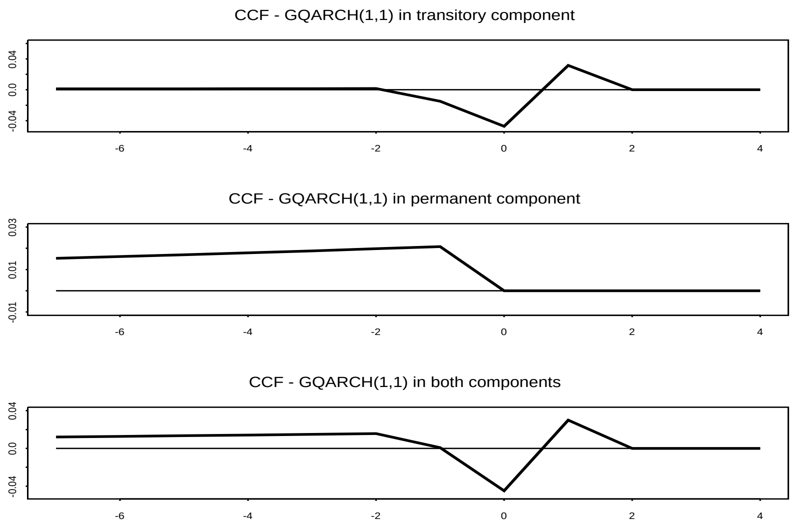


Figure 2: Cross-correlation function of $(\Delta y_t)^2(\Delta y_{t-\tau})$ of different Q-STARCH models with $q = 1$.

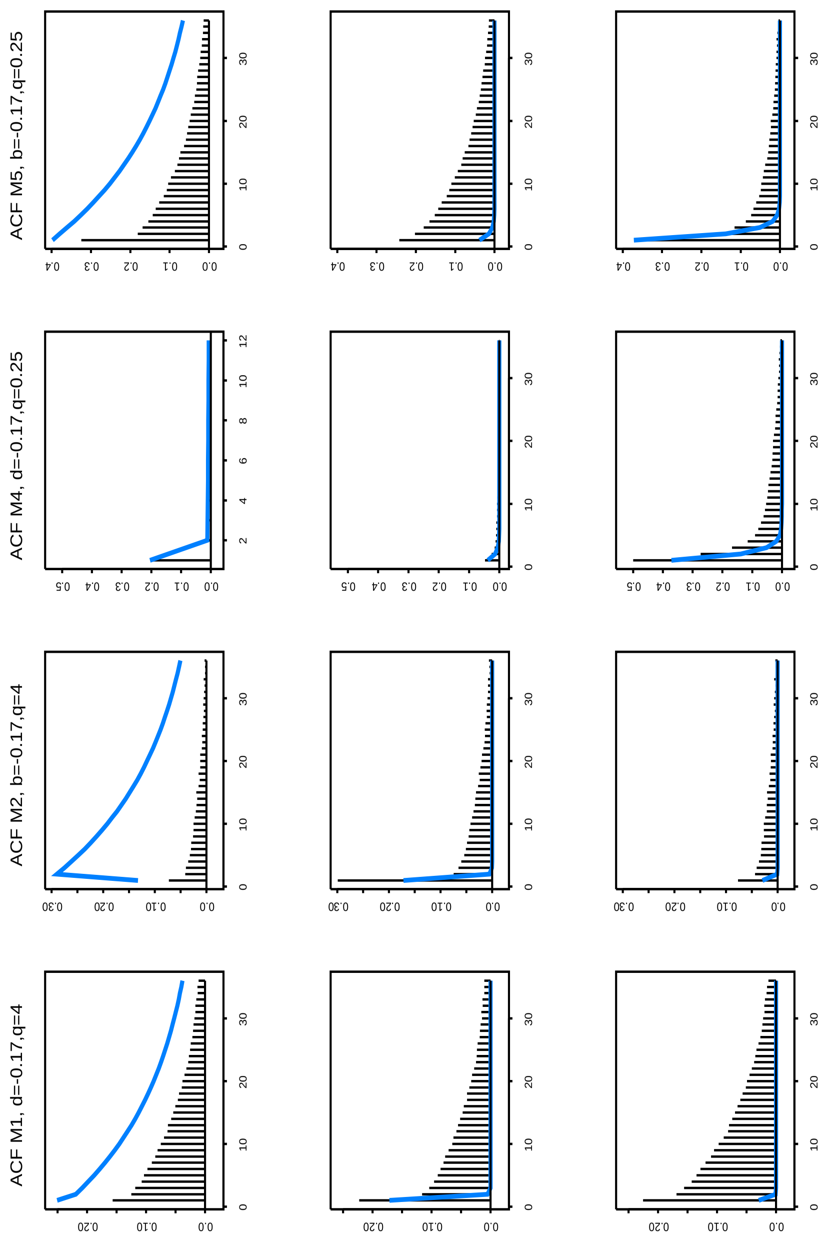


Figure 3: Mean autocorrelation function of squared series of (by rows) Δy_t, $\hat{\varepsilon}_t$ and $\hat{\eta}_t$ for a STARCH models (by columns) M1, M2, M4 and M5, with heteroscedasticity in the permanent component. Results based on 1,000 replications of series with sample size $T = 1,000$.

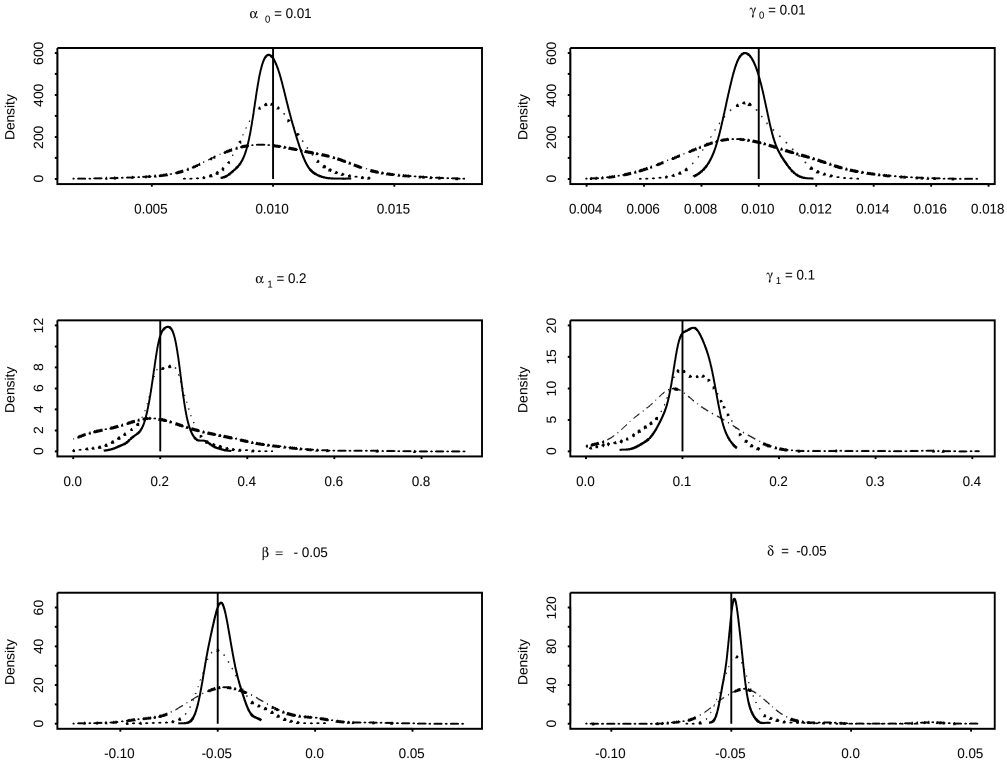


Figure 4: Kernel densities for the estimated parameters in a Q-STARCH model with asymmetry in both components. The solid line corresponds to $T = 3,000$, the dotted line to $T = 1,000$ and the dash-dotted line to $T = 300$. Parameter values are: $\alpha_0 = 0.01$, $\gamma_0 = 0.01$, $\alpha_1 = 0.2$, $\gamma_1 = 0.1$, $\beta = -0.05$ and $\delta = -0.05$.

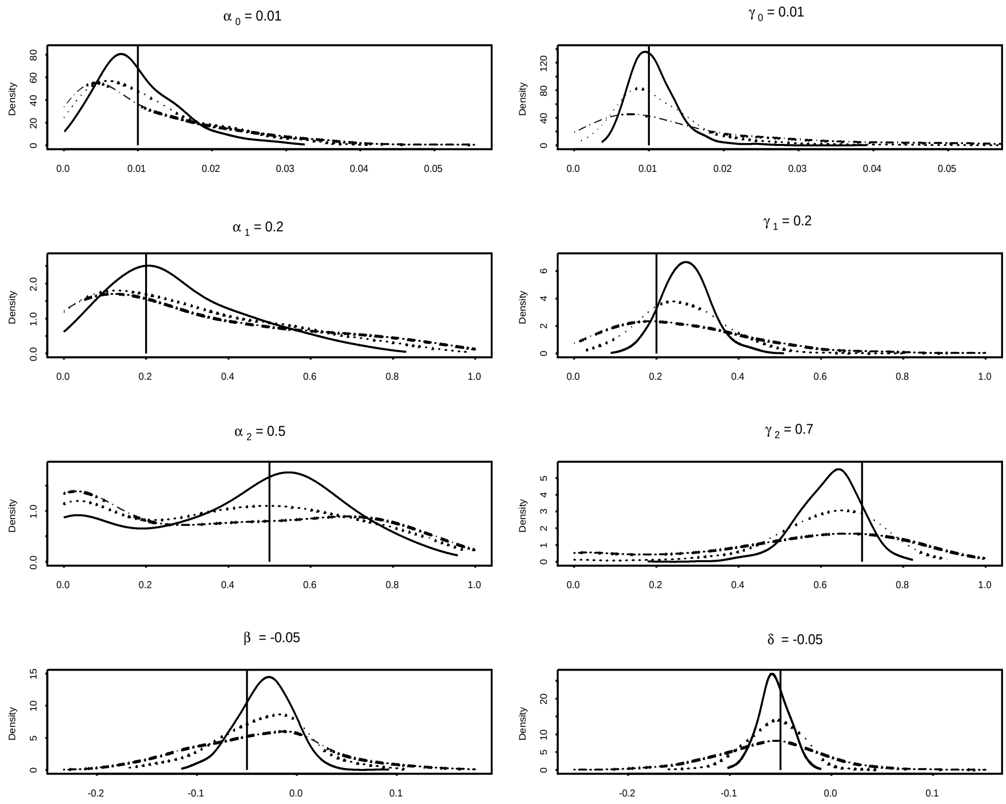
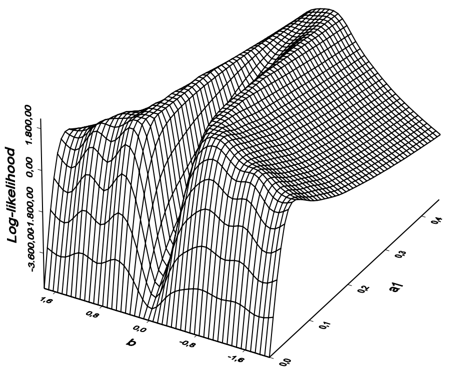
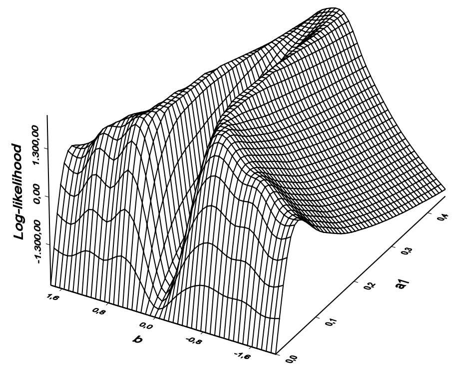


Figure 5: Kernel densities for the estimated parameters in a Q-STARCH model with asymmetry in both components. The solid line corresponds to $T = 3,000$, the dotted line to $T = 1,000$ and the dash-dotted line to $T = 300$. Parameter values are: $\alpha_0 = 0.01$, $\gamma_0 = 0.01$, $\alpha_1 = 0.2$, $\gamma_1 = 0.2$, $\alpha_2 = 0.5$, $\gamma_2 = 0.7$, $\beta = -0.05$ and $\delta = -0.05$.

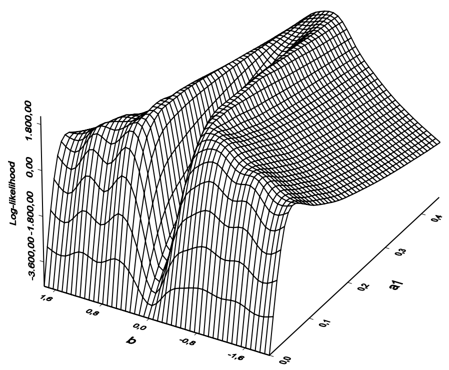
(a)



(b)



(c)



(d)

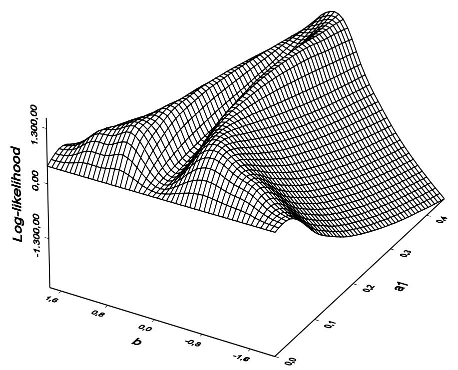


Figure 6: Likelihood with respect to a_1 and b when (a) η_t is homoscedastic, (b) η_t is an ARCH(1) process, (c) η_t is an QARCH process and (d) η_t is an GQARCH(1,1) process.

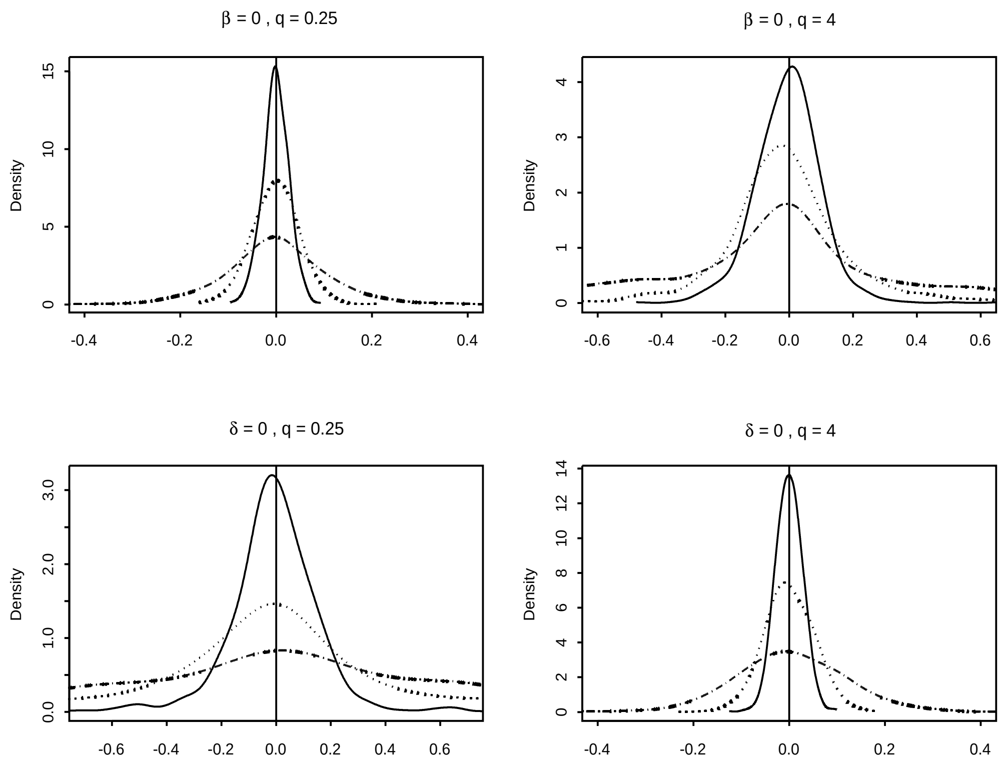


Figure 7: Kernel densities for the estimated asymmetry parameters in four Q-STARCH models. The solid line corresponds to $T = 3,000$, the dotted line to $T = 1,000$ and the dash-dotted line to $T = 300$.

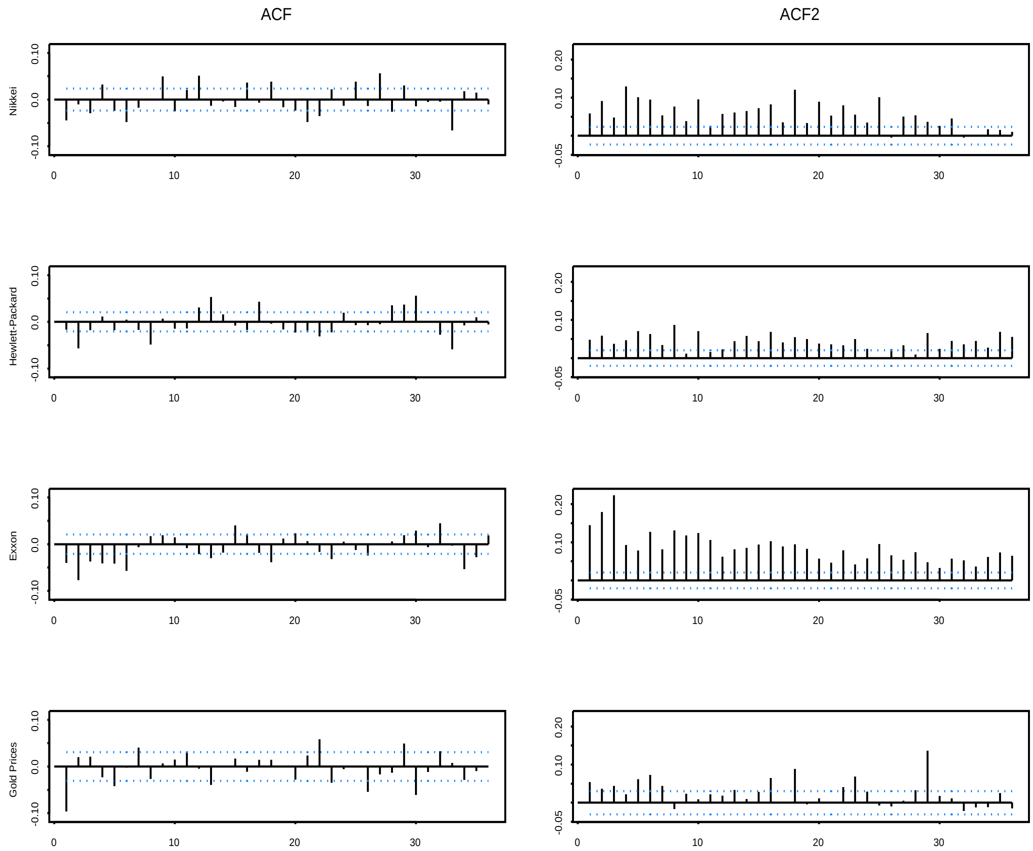


Figure 8: Sample autocorrelations of Δy_t and $(\Delta y_t)^2$ for Nikkei 225, Hewlett-Packard, Exxon and gold prices

		$T=300$	$T=1000$	$T=3000$			$T=300$	$T=1000$	$T=3000$
M1	$\rho_{(\Delta y)^2}(1)$=0.251	0.144 (0.097)	0.156 (0.088)	0.187 (0.076)	M4	$\rho_{(\Delta y)^2}(1)$=0.204	0.188 (0.072)	0.200 (0.045)	0.201 (0.027)
	$\rho_{(\Delta y)^2}(10)$=0.145	0.041 (0.071)	0.074 (0.068)	0.098 (0.058)		$\rho_{(\Delta y)^2}(10)$=0.008	0.000 (0.057)	0.005 (0.036)	0.006 (0.023)
	$\rho_{\varepsilon^2}(1)$	0.197 (0.092)	0.222 (0.067)	0.237 (0.056)		$\rho_{\varepsilon^2}(1)$	0.038 (0.065)	0.047 (0.040)	0.050 (0.028)
	$\rho_{\varepsilon^2}(10)$	0.038 (0.077)	0.064 (0.061)	0.082 (0.049)		$\rho_{\varepsilon^2}(10)$	0.003 (0.057)	0.007 (0.036)	0.009 (0.023)
	$\rho_{\eta^2}(1)$	0.181 (0.102)	0.225 (0.083)	0.255 (0.071)		$\rho_{\eta^2}(1)$	0.455 (0.098)	0.500 (0.072)	0.530 (0.063)
	$\rho_{\eta^2}(10)$	0.063 (0.079)	0.097 (0.069)	0.119 (0.058)		$\rho_{\eta^2}(10)$	0.026 (0.076)	0.052 (0.067)	0.073 (0.056)
M2	$\rho_{(\Delta y)^2}(1)$=0.132	0.046 (0.080)	0.073 (0.073)	0.091 (0.065)	M5	$\rho_{(\Delta y)^2}(1)$=0.397	0.271 (0.123)	0.324 (0.095)	0.352 (0.074)
	$\rho_{(\Delta y)^2}(10)$=0.192	0.013 (0.055)	0.025 (0.045)	0.033 (0.037)		$\rho_{(\Delta y)^2}(10)$=0.252	0.066 (0.081)	0.104 (0.070)	0.126 (0.059)
	$\rho_{\varepsilon^2}(1)$	0.252 (0.115)	0.299 (0.111)	0.257 (0.058)		$\rho_{\varepsilon^2}(1)$	0.200 (0.106)	0.242 (0.087)	0.268 (0.069)
	$\rho_{\varepsilon^2}(10)$	0.023 (0.073)	0.041 (0.058)	0.050 (0.044)		$\rho_{\varepsilon^2}(10)$	0.072 (0.081)	0.109 (0.072)	0.131 (0.059)
	$\rho_{\eta^2}(1)$	0.054 (0.089)	0.065 (0.039)	0.076 (0.075)		$\rho_{\eta^2}(1)$	0.340 (0.106)	0.355 (0.068)	0.360 (0.045)
	$\rho_{\eta^2}(10)$	0.013 (0.070)	0.024 (0.032)	0.026 (0.048)		$\rho_{\eta^2}(10)$	0.027 (0.074)	0.047 (0.054)	0.061 (0.045)
M3	$\rho_{(\Delta y)^2}(1)$=0.263	0.117 (0.107)	0.162 (0.096)	0.196 (0.083)	M6	$\rho_{(\Delta y)^2}(1)$=0.377	0.262 (0.119)	0.312 (0.093)	0.336 (0.076)
	$\rho_{(\Delta y)^2}(10)$=0.166	0.045 (0.068)	0.076 (0.064)	0.100 (0.055)		$\rho_{(\Delta y)^2}(10)$=0.237	0.063 (0.079)	0.099 (0.069)	0.118 (0.057)
	$\rho_{\varepsilon^2}(1)$	0.242 (0.094)	0.271 (0.070)	0.285 (0.057)		$\rho_{\varepsilon^2}(1)$	0.181 (0.101)	0.225 (0.084)	0.248 (0.067)
	$\rho_{\varepsilon^2}(10)$	0.056 (0.079)	0.085 (0.061)	0.102 (0.047)		$\rho_{\varepsilon^2}(10)$	0.069 (0.078)	0.105 (0.069)	0.124 (0.056)
	$\rho_{\eta^2}(1)$	0.173 (0.101)	0.213 (0.080)	0.240 (0.068)		$\rho_{\eta^2}(1)$	0.433 (0.108)	0.475 (0.076)	0.499 (0.065)
	$\rho_{\eta^2}(10)$	0.065 (0.077)	0.097 (0.064)	0.116 (0.052)		$\rho_{\eta^2}(10)$	0.478 (0.080)	0.080 (0.067)	0.102 (0.053)

Table 1: Monte Carlo results on the estimated $Corr[(\Delta y_t)^2, (\Delta y_{t-\tau})^2]$, $Corr[\hat{\varepsilon}_t^2, \hat{\varepsilon}_{t-\tau}^2]$ and $Corr[\hat{\eta}_t^2, \hat{\eta}_{t-\tau}^2]$.

	T=300	T=1000	T=3000		T=300	T=1000	T=3000
	0.0101	0.0100	0.0100		0.0112	0.0110	0.0101
α_0=0.01	(0.0023)	(0.0011)	(0.0007)	α_0=0.01	(0.0101)	(0.0080)	(0.0057)
	[0.0030]	[0.0016]	[0.0009]		[0.0243]	[0.0128]	[0.0071]
	0.2084	0.2087	0.2139		0.3158	0.2968	0.2822
α_1=0.2	(0.1328)	(0.0563)	(0.0378)	α_1=0.2	(0.2631)	(0.2282)	(0.1712)
	[0.2036]	[0.1037]	[0.0582]		[0.6679]	[0.3706]	[0.2093]
					0.3647	0.3815	0.4220
α_2=0.0				α_2=0.5	(0.3171)	(0.2894)	(0.2517)
					[1.1964]	[0.6342]	[0.3518]
	-0.0433	-0.0464	-0.0480		-0.0354	-0.0340	-0.0315
β=-0.05	(0.0246)	(0.0122)	(0.0063)	β=-0.05	(0.0676)	(0.0471)	(0.0270)
	[0.0300]	[0.0143]	[0.0085]		[0.0995]	[0.0509]	[0.0284]
	0.0096	0.0096	0.0096		0.0167	0.0121	0.0106
γ_0=0.01	(0.0020)	(0.0011)	(0.0006)	γ_0=0.01	(0.0165)	(0.0076)	(0.0034)
	[0.0035]	[0.0019]	[0.0011]		[0.0087]	[0.0044]	[0.0026]
	0.0988	0.1045	0.1082		0.2732	0.2684	0.2728
γ_1=0.1	(0.0464)	(0.0319)	(0.0188)	γ_1=0.2	(0.1782)	(0.1065)	(0.0591)
	[0.2737]	[0.1483]	[0.0849]		[0.1351]	[0.0727]	[0.0451]
					0.5365	0.6042	0.6190
γ_2=0.0				γ_2=0.7	(0.2585)	(0.1503)	(0.0771)
					[0.1970]	[0.0986]	[0.0596]
	-0.0426	-0.0469	-0.0486		-0.0605	-0.0565	-0.0564
δ=-0.05	(0.0177)	(0.0076)	(0.0033)	δ=-0.05	(0.0543)	(0.0285)	(0.0149)
	[0.0383]	[0.0193]	[0.0112]		[0.0495]	[0.0247]	[0.0137]

Table 2: Monte Carlo results for estimated parameters of Q-STARCH models with asymmetry in both components. Standard deviations in brackets. Asymptotic Standard deviation in squared brackets.

	NIK		HPQ		XOM		GOLD	
	Δy_t		Δy_t		Δy_t		Δy_t	
Mean	-0.012		0.025		0.034		0.000	
SK	0.013		-0.062		0.071		0.698*	
κ	5.588*		6.920*		4.535*		9.159*	
$\rho(\tau)$	Δy_t	$(\Delta y_t)^2$	Δy_t	$(\Delta y_t)^2$	Δy_t	$(\Delta y_t)^2$	Δy_t	$(\Delta y_t)^2$
1	-0.045*	0.059*	-0.017	0.048*	-0.040*	0.145*	-0.096*	0.054*
2	-0.010	0.091*	-0.057*	0.059*	-0.077*	0.180*	0.020	0.036*
3	-0.029*	0.048*	-0.018	0.038*	-0.037*	0.224*	0.021	0.044*
4	0.032*	0.129*	0.011	0.047*	-0.041*	0.093*	-0.023	0.022
5	-0.024	0.101*	-0.018	0.071*	-0.042*	0.078*	-0.042*	0.062*
Q(10)	18.982	127.27	17.017	76.406	39.018	444.09	16.224	20.172

Table 3: Summary statistics of (Δy_t) for Nikkei 225, Hewlett-Packard, Exxon and gold prices series.

	NIK		**HPQ**		**XOM**		**GOLD**	
$\hat{\sigma}^2_\varepsilon$	0.111		0.216		0.146		8.705 E-06	
$\hat{\sigma}^2_\eta$	1.482		7.352		1.963		8.225 E-05	
$\hat{q}$	13.288		33.975		13.411		9.448	
	ν_t		ν_t		ν_t		ν_t	
Mean	-0.018		0.011		0.026		0.014	
Std. Dev.	0.997		0.998		0.998		0.997	
SK	-0.149		0.050		0.056		0.214	
κ	4.629		4.988		3.779		5.493	
Q(10)	15.892		20.898		35.829*		4.505	
Q_2*(10)*	145.78*		193.54*		231.79*		52.913*	
	$\hat{\varepsilon}_{t/T}$		$\hat{\varepsilon}_{t/T}$		$\hat{\varepsilon}_{t/T}$		$\hat{\varepsilon}_{t/T}$	
Mean	0.000		0.001		0.000		0.001	
Std. Dev.	0.999		1.001		0.999		1.001	
SK	-0.183		-0.058		-0.034		0.190	
κ	4.318		4.778		3.833		4.136	
$\rho(\tau)$	$\hat{\varepsilon}_{t/T}$	$\hat{\varepsilon}^2_{t/T}$	$\hat{\varepsilon}_{t/T}$	$\hat{\varepsilon}^2_{t/T}$	$\hat{\varepsilon}_{t/T}$	$\hat{\varepsilon}^2_{t/T}$	$\hat{\varepsilon}_{t/T}$	$\hat{\varepsilon}^2_{t/T}$
1	-0.459*	0.332*	-0.456*	0.319*	-0.424*	0.213*	-0.463*	0.294*
2	-0.023*	0.065*	-0.060*	0.078*	-0.097*	0.156*	-0.040*	0.186*
3	-0.048*	0.081*	0.001	0.108*	0.022*	0.140*	0.034*	0.096*
4	0.044*	0.161*	0.035*	0.088*	0.007	0.085*	-0.030	0.092*
5	0.008	0.151*	-0.040*	0.091*	-0.002	0.073*	-0.015	0.130*
Q(10)	408.67*	401.70*	515.39*	431.70*	452.15*	317.46*	237.30*	208.08*
	$\hat{\eta}_{t/T}$		$\hat{\eta}_{t/T}$		$\hat{\eta}_{t/T}$		$\hat{\eta}_{t/T}$	
Mean	-0.019		0.011		0.027		0.015	
Std. Dev.	0.997		0.998		0.998		1.003	
SK	-0.141		0.051		0.056		0.191	
κ	4.546		4.966		3.828		5.393	
$\rho(\tau)$	$\hat{\eta}_{t/T}$	$\hat{\eta}^2_{t/T}$	$\hat{\eta}_{t/T}$	$\hat{\eta}^2_{t/T}$	$\hat{\eta}_{t/T}$	$\hat{\eta}^2_{t/T}$	$\hat{\eta}_{t/T}$	$\hat{\eta}^2_{t/T}$
1	0.066*	0.089*	0.029*	0.079*	0.065*	0.147*	0.095*	0.094*
2	-0.011	0.094*	-0.056*	0.081*	-0.077*	0.092*	0.024	0.078*
3	-0.043*	0.042*	-0.024*	0.090*	-0.038*	0.136*	0.029	0.062*
4	0.015	0.143*	0.006	0.103*	-0.039*	0.092*	-0.026	0.070*
5	-0.010	0.067*	-0.032*	0.066*	-0.053*	0.067*	-0.026	0.066*
Q(10)	23.805*	148.81*	22.872	196.73*	50.087*	227.56*	14.679	58.811*

Table 4: QML estimates of the parameters of the random walk plus noise model and summary statistics of estimated innovations and auxiliary residuals for the Nikkei 225, Hewlett-Packard, Exxon and gold prices time series.

	NIK	**HPQ**	**XOM**	**GOLD**
$\hat{\alpha}_0$	0.0799	0.3997	0.0801	$1.0E-05$
	(2.3963)	(3.1869)	(2.3778)	(2.3395)
$\hat{\gamma}_0$	0.0358	0.0335	0.0292	$4.0E-06$
	(3.4682)	(2.8963)	(3.3982)	(2.0326)
$\hat{\gamma}_1$	0.0824	0.0212	0.0694	0.0542
	(5.0822)	(4.8864)	(5.8808)	(2.8690)
$\hat{\gamma}_2$	0.9006	0.9737	0.9168	0.8973
	(51.7524)	(193.2517)	(71.0521)	(28.2655)
$\hat{\delta}$	-0.1087	-0.0532	-0.0598	0.0008
	(-4.9498)	(-2.2747)	(-2.7512)	(3.0712)

Table 5: QML estimates of the Q-STARCH model for the Nikkei 225, Hewlett-Packard, Exxon and gold price series.

	GER		**ITA**		**JAP**		**UK**	
	Δy_t		Δy_t		Δy_t		Δy_t	
Mean	0.000		0.000		0.000		0.000	
Std. Dev.	0.291		0.267		0.512		0.304	
SK	-0.144		-0.049		-0.055		0.117	
κ	3.384*		6.043*		3.985*		3.624*	
$\rho(\tau)$	Δy_t	$(\Delta y_t)^2$	Δy_t	$(\Delta y_t)^2$	Δy_t	$(\Delta y_t)^2$	Δy_t	$(\Delta y_t)^2$
1	-0.419*	0.183*	-0.371*	0.139*	-0.549*	0.384*	-0.376*	0.211*
2	-0.071*	0.091*	0.053*	0.092*	0.035	0.133*	-0.115*	0.028
3	0.050*	0.012	-0.090*	0.118*	0.075*	0.064*	0.107*	0.043
4	-0.053*	0.009	-0.061*	0.174*	-0.098*	0.131*	-0.105*	-0.068*
5	-0.013	0.002	-0.060*	0.070*	0.067*	0.182*	0.006	0.006
Q(10)	93.890*	23.967*	85.340*	57.551*	134.480*	114.040*	93.756*	27.164*

Table 6: Summary statistics of (Δy_t) for Germany, Italy, Japan and United Kingdom inflation series.

	GER		ITA		JAP		UK	
$\hat{\sigma}^2_\varepsilon$	0.0479		0.0329		0.1180		0.0436	
$\hat{\sigma}^2_\eta$	0.0007		0.0055		0.0010		0.0064	
$\hat{q}$	0.0149		0.1685		0.0087		0.1489	
	ν_t		ν_t		ν_t		ν_t	
Mean	0.002		-0.012		-0.092		0.002	
Std. Dev.	0.997		0.989		0.988		0.986	
SK	0.398		0.708		0.091		0.231	
κ	3.695		5.207		3.650		3.965	
Q(10)	31.865*		35.802*		14.073		11.516	
Q_2*(10)*	13.493		66.291*		46.336*		9.703	
	$\hat{\varepsilon}_{t/T}$		$\hat{\varepsilon}_{t/T}$		$\hat{\varepsilon}_{t/T}$		$\hat{\varepsilon}_{t/T}$	
Mean	0.000		0.000		0.000		0.000	
Std. Dev.	0.997		0.995		0.995		0.992	
SK	0.484		0.593		0.215		0.280	
κ	4.005		5.165		3.803		4.001	
$\rho(\tau)$	$\hat{\varepsilon}_{t/T}$	$\hat{\varepsilon}^2_{t/T}$	$\hat{\varepsilon}_{t/T}$	$\hat{\varepsilon}^2_{t/T}$	$\hat{\varepsilon}_{t/T}$	$\hat{\varepsilon}^2_{t/T}$	$\hat{\varepsilon}_{t/T}$	$\hat{\varepsilon}^2_{t/T}$
1	0.071*	0.057*	-0.046*	0.114*	-0.167*	0.159*	-0.094*	-0.011
2	-0.063*	0.023	-0.070*	0.156*	-0.044	0.087*	-0.173*	0.082*
3	-0.063*	0.013	-0.168*	0.121*	0.016	0.028	-0.023	0.037
4	-0.158*	0.091*	-0.153*	0.102*	-0.102*	0.037	-0.141*	-0.044
5	-0.157*	0.047*	-0.132*	0.123*	-0.007	0.104*	-0.060*	-0.024
Q(10)	44.175*	13.662	47.581*	63.809*	20.897	41.696*	34.726*	12.179
	$\hat{\eta}_{t/T}$		$\hat{\eta}_{t/T}$		$\hat{\eta}_{t/T}$		$\hat{\eta}_{t/T}$	
Mean	0.003		-0.024		-0.437		-0.003	
Std. Dev.	0.994		1.001		0.898		1.000	
SK	-0.060		0.464		-0.225		0.004	
κ	2.544		4.355		3.213		3.269	
$\rho(\tau)$	$\hat{\eta}_{t/T}$	$\hat{\eta}^2_{t/T}$	$\hat{\eta}_{t/T}$	$\hat{\eta}^2_{t/T}$	$\hat{\eta}_{t/T}$	$\hat{\eta}^2_{t/T}$	$\hat{\eta}_{t/T}$	$\hat{\eta}^2_{t/T}$
1	0.882*	0.724*	0.660*	0.493*	0.872*	0.787*	0.674*	0.427*
2	0.748*	0.476*	0.350*	0.190*	0.788*	0.693*	0.412*	0.131*
3	0.629*	0.298*	0.094*	0.142*	0.714*	0.622*	0.269*	0.161*
4	0.524*	0.149*	-0.054*	0.118*	0.638*	0.536*	0.135*	0.101*
5	0.457*	0.052*	-0.099*	0.114*	0.584*	0.483*	0.092*	0.042
Q(10)	1549.3*	418.10*	282.07*	184.54*	1460.8*	993.21*	354.47*	118.81*

Table 7: QML estimates of the parameters of the random walk plus noise model and and summary statistics of estimated innovations and auxiliary residuals for Germany, Italy, Japan and United Kingdom inflation series.

	GER	**ITA**	**JAP**	**UK**
$\hat{\alpha}_0$	8.245E-04	1.188E-03	5.904E-03	4.408E-04
	(1.1895)	(2.5163)	(2.0432)	(0.9132)
$\hat{\alpha}_1$	0.0321	0.2427	0.1871	0.0353
	(1.7351)	(4.3787)	(3.3198)	(1.7234)
$\hat{\alpha}_2$	0.9525	0.7463	0.7842	0.9561
	(34.6398)	(13.2465)	(12.5897)	(35.5690)
$\hat{\beta}$	0.0102	0.0122	0.0299	0.0078
	(2.0336)	(1.0003)	(1.5427)	(1.3710)
$\hat{\gamma}_0$	7.291E-04	3.988E-03	7.328E-04	3.505E-03
	(2.6102)	(4.5425)	(1.9992)	(4.3063)

Table 8: QML estimates of the Q-STARCH model for Germany, Italy, Japan and United Kingdom inflation series.